

b -PARTS OF REAL NUMBERS AND THEIR GENERALIZATION

by

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IN THE NAME OF GOD

*To all scientists and persons who try to extend the
sciences and make a world for better conciliatory living*

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List of Symbols and Abbreviations

Set of numbers

\mathbb{Z}	integers
\mathbb{N}^*	natural numbers (positive integers; \mathbb{Z}^+)
\mathbb{N}	non-negative integers
\mathbb{Z}_e	even integers
\mathbb{Z}_o	odd integers
\mathbb{Q}	rational numbers
\mathbb{Q}^c	irrational numbers
\mathbb{R}	real numbers
\mathbb{Z}_n	least nonnegative residues modulo n
$\overline{\mathbb{Z}}_n$	least residue system modulo n
\mathbb{D}_n	$\{0, \frac{1}{n}, \frac{2}{n} \dots, \frac{n-1}{n}\}$
\mathbb{R}_b	$b[0, 1) = \{bd 0 \leq d < 1\}$
$\overline{\mathbb{R}}_b$	$\mathbb{R}_b \cup \{b\} = b[0, 1]$
\mathbb{Q}_r	$\mathbb{R}_r \cap \mathbb{Q}$, where $r \in \mathbb{Q}$

Special real and arithmetic functions

$[]_b$	b -integer part function
$()_b$	b -decimal part function
$[a]_-$	various-integer part function of a
$(a)_-$	various-decimal part function of a
$\text{sgn}(x)$	signum function
$\chi_{[\alpha, \beta)}$	characteristic function of $[\alpha, \beta)$

$d(n)$	number of all positive divisors of n
$\sigma(n)$	sum of all positive divisors of n
$\Lambda(n)$	Mangoldt function
$\zeta(s)$	Riemann zeta function
Elementary and analytic number-theoretic notations	
$a c, a \nmid c$	divide (does not divide)
$a \equiv c \pmod{b}$	congruence
$\gcd(a, c)$	greatest common divisor
$[x]$	integer (integral) part of real number x
$(x), \{x\}$	decimal (fractional) part of real number x
$[x]_b$	b -integer part of real number x
$(x)_b$	b -decimal part of real number x
$f = O(g)$	$\frac{f}{g}$ is bounded (Landau symbol)
$f(x) \sim g(x)$	$\frac{f}{g} \rightarrow 1$
$\sum_{n \leq x} f(n)$	partial summation
$A([\alpha, \beta]; N; \omega)$	number of terms $x_n, 1 \leq n \leq N$, for which $(x_n) \in [\alpha, \beta)$
Real sequences	
$\{a_n\}_{n \geq 1}$	sequence
$\{a_n\}_{+\infty}^{-\infty}$	two-sided sequence
$\{a_n\}_0^N$	finite sequence
$\text{dgt}_{n,b}(a)$	b -digital sequence
$\text{dgt}_{n,b}^*(a)$	finite b -bounded sequence
Abbreviations	
u.d mod 1	uniform distribution modulo 1
u.d mod b	uniform distribution modulo b

Algebraic notations

\cdot	binary operation (multiplication)
$+$	commutative binary operation (addition)
\cdot_f	f -multiplication of \cdot
$+_b$	b -addition
(X, \cdot)	binary system
(S, \cdot)	semigroup
(G, \cdot)	group
$(G, +)$	Abelian group
f_ℓ^-	left algebraic inverse of (function) f
f_r^-	right algebraic inverse of (function) f
f^-	algebraic inverse of (function) f
f^*	left $*$ -conjugate of f
f_*	right $*$ -conjugate of f
P_Δ	left projection
P_Ω	right projection
$f \cdot g$	multiplication of functions f and g
$f + g$	addition of f and g
x_ℓ^{-1}	left inverse of x
x_r^{-1}	right inverse of x
x^{-1}	inverse of x
AB	product of A and B (subsets of a binary system)
$A + B$	sum of A and B (subsets of a commutative binary system)
$A \cdot B$	direct product of A and B
$A \dot{+} B$	direct sum of A and B
$A : B$	standard direct product of A and B
$A \ddot{+} B$	standard direct sum of A and B
$A - B$	subtraction of A and B
$\Delta \backslash G$	right co-sets of Δ in G
Δ / G	left co-sets of Δ in G
$\frac{G}{H}$	factor group
$H \leq K$	H is subgroup of K

$H \leq K$	H is sub-semigroup of K (K is group or semigroup)
$H \cong K$	H is isomorphic to K
$\langle b \rangle$	cyclic subgroup generated by b
Ω_ℓ^{-1}	$\{\omega_\ell^{-1} \mid \omega \in \Omega\}$
Ω_r^{-1}	$\{\omega_r^{-1} \mid \omega \in \Omega\}$
Ω^{-1}	$\{\omega^{-1} \mid \omega \in \Omega\}$
Other notations	
$[R]_b$	$\{[r]_b \mid r \in R\}$
$(R)_b$	$\{(r)_b \mid r \in R\}$
R_b	b -bounded real set
2^A	power set of A
$ A $	cardinal number of A
$A \setminus B$	set-theoretic difference of A and B
$\iota = \iota_X$	identity function (on X)
fg	composition of functions f and g
f^n	n -composition of f
$f _A$	f restricted to A
f^{-1}	inverse function

List of Publications and Presentations

M.H.Hooshmand and H. Kamarul Haili, *Some Algebraic Properties of b -Parts of Real Numbers*, Šiauliai Math. Semin., Vol.3, No.11 (2008), 115-121.

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BAHAGIAN-B NOMBOR NYATA DAN PENGITLAKANNYA

Abstrak

Tiga aspek iaitu peringkat permulaan, analisis dan kaedah algebra dikaji dengan mempertimbangkan ciri-ciri asas dan utama bagi bahagian- b nombor nyata.

Bagi aspek pertama, terdapat beberapa subjek dan aplikasi seperti:

- (a) Penjelasan teori nombor bagi bahagian- b dan ciri-ciri asasnya.
- (b) Aplikasi bahagian- b untuk perwakilan unik nombor nyata terhingga dan tidak terhingga, penggunaannya dan algoritma pembahagian itlak; beberapa rumus langsung untuk digit bagi pengembangan unik nombor nyata tidak terhingga terhadap asas integer diperkenalkan serta perwakilan unik nombor nyata terhingga terhadap asas sebarang (nombor nyata $0, \pm 1$) dibuktikan.
- (c) Aplikasi bahagian- b untuk menentukan bentuk am subset nyata bagi kala- b dan fungsinya.

Bagi aspek analisis, beberapa rumus asimtot dan untuk hasil tambah separa bahagian- b serta baki untuk pembahagian itlak nombor nyata positif dipertimbangkan. Konsep taburan seragam jujukan nyata modulo- b (untuk sebarang nombor nyata $b \neq 0$) dengan menggunakan fungsi bahagian perpuluhan- b juga akan dijelaskan sebagai aplikasi untuk bahagian- b .

Untuk aspek ketiga, penambahan- b nombor nyata iaitu bahagian perpuluhan- b bagi penambahan biasa dan "kumpulan nyata baki- b terkecil" (kumpulan terbatas- b) diperkenalkan dan ciri-ciri serta hubungan dengan kumpulan nyata dan kumpulan nisbah dikaji.

Pengitlakan bahagian- b untuk sebarang kumpulan juga merupakan topik lain yang

dikaji. Melalui cara ini, dapat ditunjukkan bagaimana suatu elemen dalam kumpulan boleh diwakilkan dengan uniknya melalui pengulangan dan bahagian- f , seperti integer- b dan bahagian perpuluhan- b nombor nyata.

Pertimbangan ini memberi suatu algoritma pembahagian itlak untuk kumpulan Abelian tanpa elemen berperingkat terhingga. Dalam bahagian ini juga, penguraian dan fungsi kalis sekutuan terhadap kumpulan difokuskan (termasuk kumpulan separuh sistem binari) dan juga penyelesaian fungsian persamaan.

B-PARTS OF REAL NUMBERS AND THEIR GENERALIZATION

Abstract

Considering the basic and main properties of the b -parts of real numbers, we have studied them in three aspects: elementary, analytic and algebraic methods.

As the first aspect we have several subjects and their applications such as:

- (a) Number theoretic explanations of b -parts and their elementary properties.
- (b) Application of b -parts for unique finite and infinite representation of real numbers, applying them and the generalized division algorithm, we not only introduce some direct formulas for digits of the unique infinite expansion of real numbers to the base an integer but also prove a (new) unique finite representation of real numbers to the base an arbitrary real number (not equal $0, \pm 1$).
- (c) Application of b -parts for determining the general form of b -periodic real subsets and functions.

As for analytic aspect, we consider some asymptotic and direct formulas for the partial summation of b -parts and remainders of the generalized divisions of a given positive real number. Also we intend to explain the conception of uniform distribution of real sequences modulo b (for an arbitrary real number $b \neq 0$) by using the b -decimal part function, as an application of the b -parts.

As the third aspect we introduce b -addition of real numbers that is b -decimal part of their ordinary addition and "the least real b -residues group" (b -bounded group) and study their properties as well as relations to real groups and their quotient groups.

Generalization of b -parts for arbitrary groups is another topic that we study. In this way we show how an element of a group can be uniquely represented by cyclic and f -part, like b -integer and b -decimal part of real numbers. This consideration will

give us the generalized division algorithm for Abelian groups with no element of the finite order. In this part of our research we also focus on decomposer and associative functions on groups (even on semi-groups and binary systems) and solve the related functional equations.

Chapter 1

Introduction

1.1 Introduction

In the real numbers system, the (integer) radix or base is usually the number of unique digits, including zero, that a numerical system uses to represent numbers. Representations of real numbers is an important topic in Number Theory and the radix representation is its most important branch. A representation can be infinite or finite, unique or non-unique, digital or non-digital. For example representation of real numbers by some functions (f-representation) or continued fractions are different from the radix and b -adic representation and may be not unique or digital.

For a given base b (that may be integer or non-integer), every representation corresponds to exactly one real number.

On the other hand the b -parts of real numbers are as a generalization of the integer and decimal parts of real numbers. They have many interesting number theoretic explanations, algebraic and analytic properties. Also the generalized division algorithm is one of their results that has several important applications especially for unique finite representation of real numbers. The b -parts have some applications for

infinite and finite radix-representations of real numbers. For instance by using them and b -digital sequences, not only the unique infinite b -representation of real numbers, based on an arbitrary integer $b \neq 0, \pm 1$, is proved but also some direct formulas for their digits are obtained.

1.2 Background of study

As is well-known any real number $a > 0$ has a series expansion to any integer base $b > 1$. Almost every $a > 0$ (with respect to the lebesgue measure) has a unique b -adic representation (as the most common and famous radix representation of real numbers) of the form

$$a = a_N a_{N-1} \cdots a_1 a_0 \cdot a_{-1} a_{-2} a_{-3} \cdots_b = \sum_{n=N}^{-\infty} a_n b^n \quad : \quad a_n \in \{0, 1, \dots, b-1\}, \quad (1.2.1)$$

(see [4] and [5]). Only rational numbers of the form $a = \frac{p}{q}$ where $\gcd(p, q) = 1$, $q = p_1^{\alpha_1} \cdots p_m^{\alpha_m}$ and p_i 's are the prime divisors of b , have two different expansions of the form (1.2.1). One of them being finite while the other expansion ends in an infinite string of $b-1$'s.

Representation of real numbers in non-integer bases has been systemically studied since 1957 by Renyi [14] and Parry [15] (1960). By the expansion (to the base b), they mean

$$x = \sum_{k=1}^{\infty} \frac{a_k}{b^k} = \sum_{k=1}^{\infty} a_k b^{-k}, \quad a_k \in \{0, 1, \dots, [b]\}, \quad (1.2.2)$$

where $x \in [0, 1)$, $b > 1$ and $[b]$ is the integer part of b .

In the integer case ($b \in \mathbb{Z}$ and $b > 1$) any real number $x \in [0, 1)$ can be extended to the base b , by (1.2.2). However for a given non-integer $b > 1$ almost every $x \in$

$[0, 1)$ has infinitely many different series expansion of the form (1.2.2) (see [8]). An expansion of the $x \in [0, 1)$ of (1.2.2) can be obtained by

$$a_k = [bT_b^{k-1}(x)] = [b(b(\cdots (bx) \cdots))],$$

where $[.]$ and $(.)$ denote the integer and decimal part of b , respectively and $T_b : [0, 1) \rightarrow [0, 1)$ is defined by $T_b(x) = (bx)$ and T_b^k is the k -fold composition of T_b (k -iteration of T_b). The first of the above equations generates b -ray expansions, dynamically (see [10]).

In the 1990's a group of Hungarian mathematicians led by Paul Erdős studied the radix expansion (1.2.2) to the non-integer base $b = q \in (1, 2)$, where $\varepsilon_k = a_k \in \{0, 1\}$ (because $[q] = 1$, $0 \leq \varepsilon_k \leq [q]$ and the representation is digital). They also considered the representation $1 = \sum_{k=1}^{\infty} \varepsilon_k b^{-k}$ (see [16]). The studies have been followed by Glendinning and Sidorov [8] (2001).

On the other hand b -parts of a , for arbitrary real numbers a and $b \neq 0$, were introduced and studied in [21]. The b -parts are as a generalization of the integer and decimal parts of real numbers and have been used for evaluating some direct formulas for the digits a_n and also a new proof of (1.2.1). Recall that for a real number a the notation $[a]$ is the largest integer not exceeding a and $(a) = a - [a]$ is the decimal part of a (also it is denoted by $\{a\}$ and is called fractional part of a). One can see so many elementary and analytic number theoretic properties of integer and fractional parts of real numbers in [1]. If b is a nonzero constant real number, then for any real number a , its b -parts are defined as follows

$$[a]_b = b\left[\frac{a}{b}\right], \quad (a)_b = b\left(\frac{a}{b}\right).$$

Here the notation $[a]_b$ is called b -integer part of a and $(a)_b$ b -decimal part of a . Also

$[a]_b$ and $(a)_b$ are called *b-parts* of a .

There are some interesting number theoretic explanations for *b-parts* that state, if $b > 0$, then $(a)_b$ is the remainder of the (generalized) division of a by b and $[a]_b$ is the largest element of $b\mathbb{Z}$ not exceeding a . Moreover if b is positive integer, then $[a]_b$ is the same unique integer of the residue class $\{[a] - b + 1, [a] - b + 2 \cdots, [a]\} \pmod{b}$ divisible by b .

The various properties of *b-parts* can be considered from the three aspects: number theoretic, analytic and algebraic senses. Also the *b-parts* functional equations

$$f(f(x) + y - f(y)) = f(x) \quad : \quad (\text{b-parts type functional equation}),$$

$$f(x + y - f(y)) = f(x) \quad : \quad (\text{strong b-parts type functional equation}),$$

$$f(f(x + y) + z) = f(x + f(y + z)) = f(x + y + z) \quad : \quad (\text{strongly associative equation}),$$

link it to some topics in functional equations.

One of the most important properties of *b-parts* functions is that the sum of their ranges makes a direct decomposition for the real numbers group. In fact for any real number $b \neq 0$

$$\mathbb{R} = b\mathbb{Z} \dot{+} b[0, 1) = \langle b \rangle \dot{+} \mathbb{R}_b, \quad (1.2.3)$$

where $\mathbb{R}_b = ()_b(\mathbb{R}) = b[0, 1) = \{bd | d \in [0, 1)\}$, $[]_b(\mathbb{R}) = [\mathbb{R}]_b = b\mathbb{Z} = \langle b \rangle$ and $\dot{+}$ denotes the direct sum. Therefore the range of the *b*-decimal part function is a direct factor of real numbers group that is not a subgroup (and also sub-semigroup), and this leads us to more studies and also a generalization of *b-parts* regarding to the direct decomposition (factorization) by subsets. The first study about factorization of a group by subsets comes from the paper [24] of Hajos (1942). In order to solve a

geometric problem, he introduced the notation of direct product of subsets. He said that the group G is the direct product of two its subsets A and B if each element of G is uniquely expressible in the form ab , $a \in A$, $b \in B$, and showed that under certain circumstances one of the set is a group. The topic has been followed by Stein 1967, [12], Sands 1991, [11], Szabo 2005, [13], and other mathematicians (see [6]).

The uniform distribution of real sequences is another branch of Number Theory that has some connections to the b -adic representation of real numbers [3, Chapter 8]. Also the b -parts were used for defining continuous uniform distributed real functions mod $m \in \mathbb{Z}$ ($m > 1$, see [3, p. 316]). The uniform distribution of real sequences mod 1 is defined by using their decimal parts, and also uniform distributed integer sequences modulo an integer $m \geq 2$, was introduced in [18]. Hence it seems that b -decimal part function will be useful for interpreting uniform distribution of real sequences mod b .

1.3 Objectives of study

The main objectives of the study are listed below:

1. To prove some asymptotic formulas for b -parts of real numbers.
2. To introduce a unique finite representation of real numbers to the base an arbitrary real number $b \neq 0, \pm 1$ (by using the b -parts and the generalized division algorithm) and prove a necessary and sufficient condition for the finite b -representation to be digital.
3. To study b -addition of real numbers and b -bounded groups and characterization of b -periodic real subsets and functions (by using the b -parts functions).
4. To generalize the conception b -parts from real numbers group to arbitrary groups.
5. To solve b -parts functional equations and their generalized forms on arbitrary

groups, semigroups and even some binary systems.

6. To generalize the topic b -bounded groups and study f -decompositional groups, where f is a b -parts type function on a group G .

1.4 Organization of thesis

The organization of the thesis is as follows:

Chapter 2 is about elementary and basic properties of b -parts of real numbers and some of their analytic properties.

Chapter 3 contains a study of infinite and finite unique b -representation of real numbers and also uniform distribution of real sequences modulo b .

In Chapter 4 we first discuss b -addition of real numbers and then study b -bounded groups and also b -periodic real subsets.

Chapter 5 starts with introducing decomposer functions on groups and goes on to consider generalization of b -parts for groups. Also the generalized division algorithm for some Abelian groups will be studied.

In chapter 6 b -parts type functional equations and their generalized forms on semigroups and binary systems will be studied.

Chapter 7 contains a generalization of b -bounded groups and a study of f -decompositional groups.

Chapter 2

b -Parts of Real Numbers and Their Analytic Properties

In this chapter we consider b -parts of real numbers and their number theoretic explanations, then study some of their properties.

2.1 Introduction

For any real number

$$(a) = a - [a] \text{ (the de}$$

For all real numbers

We call the notation

$$]_b$$

and $(a)_b$ are called

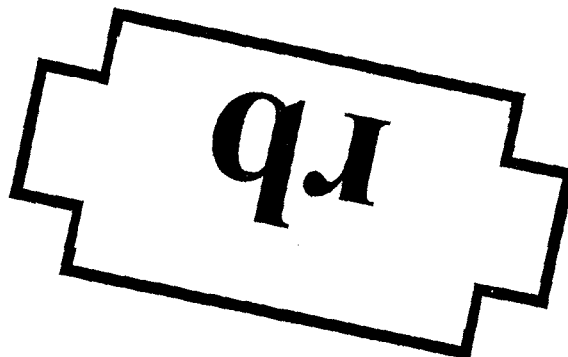
Since $(a)_1 = ($

tion,

sometimes we

Now let a, t

where



q, r are integers and $0 \leq r < b$, so

$$(a)_b = (bq + r)_b = (r)_b = r.$$

It means that $(a)_b$ is the same remainder of the division of a by b . It is an important fact that leads us to several properties of b -parts.

2.2 Elementary properties of b -parts of real numbers

This section introduces some basic properties of b -parts which will be used repeatedly.

Proposition 2.2.1. *The following properties hold, for every real number a :*

(I) *For every $\beta \in b\mathbb{Z}$, we have $[a + \beta]_b = [a]_b + \beta$, $(a + \beta)_b = (a)_b$ so if m, n are integers, then*

$$(ma + nc)_b = (m(a)_b + nc)_b = (ma + n(c)_b)_b = (m(a)_b + n(c)_b)_b = ((ma + nc)_b)_b. \quad (2.2.1)$$

$$(II) \quad (a)_b = a \iff a \in b[0, 1) \iff [a]_b = 0,$$

$$(III) \quad (a)_b = 0 \iff a \in b\mathbb{Z} \iff [a]_b = a,$$

(IV)

$$|(a)_b| < |b|, \quad \frac{[a]_b}{\text{sgn}(b)} \leq \frac{a}{\text{sgn}(b)} < \frac{[a]_b + b}{\text{sgn}(b)}, \quad |a| - |b| < |[a]_b|,$$

where sgn is the signum function. So if $b > 0$, then $0 \leq (a)_b < b$ and $[a]_b - b < [a]_b \leq a$.

(V) *If b is a positive integer, then*

$$([a])_b = [(a)_b] = (a)_b - (a) = (a)_b - ((a)_b) = [a] - [[a]]_b = [a] - [a]_b. \quad (2.2.2)$$

(VI) For every real number $b \neq 0$, the set $\{(na)_b | n \in \mathbb{Z}\}$ is finite if and only if $a \in b\mathbb{Q}$ (i.e. $\frac{a}{b}$ is rational number). In addition if $\frac{a}{b}$ is irrational, then the sequence $(na)_b$ is dense in the close interval $b[0, 1]$ ($= [0, b]$ or $[b, 0]$).

Proof. Let $\beta = kb$ where $k \in \mathbb{Z}$. Then

$$[a + \beta]_b = b[\frac{a}{b} + \frac{\beta}{b}] = b[\frac{a}{b} + k] = b[\frac{a}{b}] + bk = [a]_b + \beta,$$

$$(a + \beta)_b = a + \beta - [a + \beta]_b = a + \beta - [a]_b - \beta = (a)_b,$$

$$(m(a)_b + nc)_b = (ma - m[a]_b + nc)_b = (ma + nc)_b,$$

because $m[a]_b \in b\mathbb{Z}$. The proof of the other parts of (2.2.1) are similar.

Now we have

$$(a)_b = a \Leftrightarrow a - [a]_b = a \Leftrightarrow [\frac{a}{b}] = 0 \Leftrightarrow 0 \leq \frac{a}{b} < 1 \Leftrightarrow a \in b[0, 1)$$

$$(a)_b = 0 \Leftrightarrow a - [a]_b = 0 \Leftrightarrow \frac{a}{b} = [\frac{a}{b}] \Leftrightarrow \frac{a}{b} \in \mathbb{Z} \Leftrightarrow a \in b\mathbb{Z}$$

To prove (IV), first we have $|(a)_b| = |b| |(\frac{a}{b})_1| < |b|$. So

$$|[a]_b| = |a - (a)_b| \geq |a| - |(a)_b| > |a| - |b|.$$

Now multiplying the inequality $[\frac{a}{b}] \leq \frac{a}{b} < [\frac{a}{b}] + 1$ by b , implies $[a]_b \leq a < [a]_b + b$ if $b > 0$ and $[a]_b + b < a \leq [a]_b$ if $b < 0$. Therefore the proof of (IV) is complete.

Considering the identities $a = [a]_b + (a)_b = [[a]]_b + ([a])_b + (a)$ and since $b \in \mathbb{Z}^+$, $([a])_b \in \mathbb{Z}$, we have $0 \leq ([a])_b \leq b - 1$ so $0 \leq ([a])_b + (a) < b$ thus $([a])_b = (a)_b - (a)$ and $[[a]]_b = [a]_b$. On the other hand

$$[a] = [[a]]_b + ([a])_b = [[a]_b + (a)_b] = [a]_b + [(a)_b].$$

Therefore we can deduce the identities (2.2.2).

Now if m and n are two distinct integers, then $(na)_b = (ma)_b$ if and only if $a =$

$\frac{[na]_b - [ma]_b}{n-m}$ (notice that $[na]_b - [ma]_b \in b\mathbb{Z}$). Also if n_0 is a fixed integer and $a = \frac{m}{n_0}b$, then $(n_0a)_b = (ma)_b = 0$ and for every integer k we have

$$\begin{aligned} (ka)_b &= ([k]_{n_0}a + (k)_{n_0}a)_b = ([\frac{k}{n_0}]n_0a + (k)_{n_0}a)_b = ([\frac{k}{n_0}](n_0a)_b + (k)_{n_0}a)_b \\ &= ((k)_{n_0}a)_b \in \{0, (a)_b, (2a)_b, \dots, ((n_0 - 1)a)_b\}. \end{aligned}$$

In fact we have $\{(na)_b | n \in \mathbb{Z}\} = \{0, (a)_b, (2a)_b, \dots, ((n_0 - 1)a)_b\}$.

Finally the identity $(na)_b = b(n\frac{a}{b})_1$ and the Kronecker's theorem (see [1]) imply that the sequence $\{(na)_b\}_{n \geq 1}$ is dense in the close interval $b[0, 1]$, if $\frac{a}{b}$ is irrational. \square

Remark 2.2.1. As can be seen from Chapter 4, in fact the set $\{(na)_b | n \in \mathbb{Z}\}$ is a cyclic subgroup of the b -bounded group $(\mathbb{R}_b, +_b)$ (the least real residues group modulo b), where $\mathbb{R}_b = b[0, 1)$ and $+_b$ is the b -addition $(x +_b y = (x + y)_b, \forall x, y \in \mathbb{R})$. The above property states that a cyclic subgroup of $(b[0, 1), +_b)$, generated by a , is dense in $b[0, 1]$ if and only if $\frac{a}{b}$ is irrational. Also if $\frac{a}{b} = \frac{m_0}{n_0}$ is a rational number for which $n_0 > 0$, $\gcd(m_0, n_0) = 1$, then the cyclic group $\langle a \rangle$ is finite and

$$\langle a \rangle = \{0, (a)_b, (2a)_b, \dots, ((n_0 - 1)a)_b\},$$

so the order of a is equal to n_0 .

If a and b are integers, then $(a)_b$ is also an integer. Hence this raises the question when is $(a)_b$ an integer?. The answer of this question is important, because first we want to know that if $a, b \in \mathbb{R}$ and $b > 0$, when is the remainder of the division of a by b an integer (like the quotient of the division) . Secondly we need it (in the next chapter) to determine when the finite b -representation of a real number is digital. Before stating the related lemma notice that:

A necessary condition for $(a)_b$ to be an integer is that $a \in \langle 1, b \rangle$ (where $\langle 1, b \rangle$ is the real subgroup generated by 1 and b). So if $(a)_b$ is integer, then the real numbers a , b and 1 are linearly dependent on \mathbb{Z} and \mathbb{Q} . The converse is not valid (the conditions are not sufficient), because if $b = \sqrt{2}$ and $a = 2\sqrt{2} + 2$, then $a \in \langle 1, b \rangle$ and a , b and 1 are linearly dependent, and $(a)_b = 2 - \sqrt{2}$. But the necessary and sufficient condition for $(a)_b$ to be an integer is that a belongs to a subset of $\langle 1, b \rangle$ as follows:

$$\{m + kb | k \in \mathbb{Z}, m \in \mathbb{Z} \cap \mathbb{R}_b\},$$

because in this case

$$(a)_b = (m + kb)_b = (m)_b = b\left(\frac{m}{b}\right)_1 = b\frac{m}{b} = m.$$

(its converse is clear). Also in general we have the following inferences:

$$a, b \in \mathbb{Q} \Rightarrow (a)_b \in \mathbb{Q}, \quad a \in \mathbb{Q}^c \text{ \& } b \in \mathbb{Q} \Rightarrow (a)_b \in \mathbb{Q}^c$$

$$a \in \mathbb{Q} \setminus \mathbb{R}_b \text{ \& } b \in \mathbb{Q}^c \Rightarrow (a)_b \in \mathbb{Q}^c.$$

In the case a and b are irrationals, if the real numbers a , b and 1 are linearly independent, then $(a)_b$ is also irrational.

Lemma 2.2.2. *If $b \neq 0$ is a rational number, then $(a)_b$ is integer if and only if a and b have the reduced rational forms $a = \frac{\alpha}{\beta}$ and $b = \frac{\gamma}{\lambda}$ (i.e. $\beta, \lambda \in \mathbb{Z}^+$ and $\gcd(\alpha, \beta) = \gcd(\gamma, \lambda) = 1$) such that*

$$\beta | \gcd(\lambda, (\alpha)_\gamma) \quad , \quad \left(\frac{\alpha}{\gamma}\right)_1 < \frac{\beta}{\lambda}.$$

Proof. If $b \in \mathbb{Q}$ and $(a)_b \in \mathbb{Z}$, then $a \in \mathbb{Q}$, clearly. So there exist integers α , γ and positive integers β , λ for which $\gcd(\alpha, \beta) = \gcd(\gamma, \lambda) = 1$ and $a = \frac{\alpha}{\beta}$, $b = \frac{\gamma}{\lambda}$. Now

putting $\theta = [\frac{a}{b}]$ we have $(a)_b = \frac{\alpha\lambda - \beta\theta\gamma}{\beta\lambda}$ thus $\beta\lambda \mid \alpha\lambda - \beta\theta\gamma$ and so $\beta \mid \lambda, \lambda \mid \beta\theta$. Therefore there exists integer d such that $[\frac{a}{b}] = [\frac{\alpha\lambda}{\beta\gamma}] = \theta = \frac{\lambda}{\beta}d$ and this implies $\frac{\alpha}{\gamma} - \frac{\beta}{\lambda} < d \leq \frac{\alpha}{\gamma}$.

But since $\frac{\beta}{\lambda} \leq 1$, then

$$\frac{\alpha}{\gamma} - \frac{\beta}{\lambda} < d = [\frac{\alpha}{\gamma}] = \frac{\alpha}{\gamma} - (\frac{\alpha}{\gamma}).$$

So $(\frac{\alpha}{\gamma})_1 < \frac{\beta}{\lambda}$ and

$$(a)_b = \frac{\alpha\lambda - \beta\theta\gamma}{\beta\lambda} = \frac{\alpha - d\gamma}{\beta} = \frac{(\alpha)_\gamma}{\beta},$$

therefore $\beta \mid \gcd(\lambda, (\alpha)_\gamma)$.

Conversely suppose that the conditions are held. Then $\beta \mid \lambda$ and $(\frac{\alpha}{\gamma})_1 < \frac{\beta}{\lambda}$ imply $[\frac{a}{b}] = [\frac{\lambda}{\beta} \frac{\alpha}{\gamma}] = \frac{\lambda}{\beta} [\frac{\alpha}{\gamma}]$ (considering the next note) and so $(a)_b = \frac{\alpha}{\beta} - \frac{\gamma}{\lambda} \frac{\lambda}{\beta} [\frac{\alpha}{\gamma}] = \frac{(\alpha)_\gamma}{\beta} \in \mathbb{Z}$. \square

Note: For every real numbers x and $\kappa \neq 0$ we have

$$[\kappa x] = \kappa[x] \Leftrightarrow (\kappa x) = \kappa(x) \Leftrightarrow (x) = (x)_{\frac{1}{\kappa}} \Rightarrow (x) < |\frac{1}{\kappa}|,$$

and the converse of the last conclusion is valid if $\kappa = k$ is a natural number ($x \in [0, \frac{1}{k}) + \mathbb{Z} \Leftrightarrow (x) < \frac{1}{k} \Leftrightarrow (x) = (x)_{\frac{1}{k}}$). So we conclude that the condition $(\frac{\alpha}{\gamma})_1 < \frac{\beta}{\lambda}$ in the above theorem can be replaced by $[\frac{\lambda}{\beta} \frac{\alpha}{\gamma}] = \frac{\lambda}{\beta} [\frac{\alpha}{\gamma}]$.

Corollary 2.2.3. Let a, b be reduced rational numbers $a = \frac{\alpha}{\beta}$ and $b = \frac{\gamma}{\lambda}$.

(i) A necessary condition on a and b for $(a)_b$ to be an integer is

$$\lambda(\frac{\alpha}{\gamma})_1 < \beta \leq \min\{\lambda, |(\alpha)_\gamma|\}.$$

Also (in that case) if $\beta \nmid \lambda$ or $\beta \nmid (\alpha)_\gamma$ or $\beta \geq |\gamma|$ or $\beta \leq \lambda(\frac{\alpha}{\gamma})_1$, then $(a)_b$ is a non-integer rational number.

(ii) If $b > 0$, then the b -bounded remainder of the (generalized) division of a by b is an integer if and only if $\beta \mid \gcd(\lambda, \text{the remainder of the division of } \alpha \text{ by } \gamma)$ and

$(\frac{\alpha}{\gamma})_1 < \frac{\beta}{\lambda}$ (notice that the identity $\frac{\alpha}{b} = \frac{\alpha\lambda}{\beta\gamma}$ implies there exists another remainder for the division a by b for which is $\beta\gamma$ -bounded and can be gotten from the ordinary division algorithm).

2.3 Number theoretic explanations of b-parts

For every positive integer b and real number a , $[a]_b$ is the same unique integer of the residue class $\{[a] - b + 1, \dots, [a]\} \pmod{b}$ that is divisible by b (because $b|[a]_b$ and $[a] - b + 1 \leq [a]_b \leq [a]$). Also, for the general explanation of $[a]_b$, if $b > 0$, then $[a]_b$ is the largest element of $b\mathbb{Z}$ not exceeding a and if $b < 0$, then $[a]_b$ is the least element of $b\mathbb{Z}$ not less than a .

Now let a, b are positive integers. By the division algorithm we have $a = bq + r$ where q, r are integers and $0 \leq r < b$, so

$$(a)_b = (bq + r)_b = (r)_b = r.$$

It means that $(a)_b$ is the same remainder of the division of a by b . This is an important fact that leads us to the generalized division algorithm (for real numbers) and algebraic properties of b -parts. Now we first state and give another proof for the generalized division algorithm, and then using it we can introduce the number theoretic explanation of b -decimal part in the general case.

Theorem 2.3.1. *Suppose $b \neq 0$ a fixed real number.*

(a) *(The unique representation of real numbers by b-parts) For every real number a there exist unique numbers a_1 and a_2 such that*

$$a = a_1 + a_2, \quad a_1 \in b\mathbb{Z}, \quad a_2 \in b[0, 1).$$

(b) (The generalized division algorithm) For every real number a , there exist a unique integer q and a unique non negative real number r such that

$$a = bq + r \quad , \quad 0 \leq r < |b|.$$

(q and r are called integer quotient and b -bounded remainder of the division of a by b , respectively.)

Proof. i) Clearly $a = [a]_b + (a)_b$. Now if $a = a_1 + a_2$ with the asserted conditions then, $(a)_b = (a_1 + a_2)_b = (a_2)_b = a_2$ so $a_1 = [a]_b$.

ii) Let $b > 0$ and put $S = \{a - bq | q \in \mathbb{Z}, a - bq \geq 0\}$. Clearly $S \neq \emptyset$ and is bounded below, now put $r = \inf S$. If q_* is an integer for which $\frac{a}{b} - 1 < q_* \leq \frac{a}{b}$, then $a - bq_* \in S \cap [0, b)$. Moreover if $a - bq_1$ and $a - bq_2$ are elements of $S \cap [0, b)$, then $-b < b(q_1 - q_2) < b$ so $q_1 = q_2$ and $r_1 = a - bq_1 = a - bq_2 = r_2$. Therefore $S \cap [0, b) = \{r\}$ and the proof, for $b > 0$, is complete.

Now if $b < 0$, then there exist $q' \in \mathbb{Z}$ and $0 \leq r < -b = |b|$ such that $a = (-b)q' + r$. Putting $q = -q'$, we have $a = bq + r$ and the conditions hold. \square

Applying the above theorem we can here state the general number theoretic explanation of $(a)_b$:

If $b > 0$, then $(a)_b$ is the same b -bounded remainder of the (generalized) division of a by b , and if $b < 0$, then $(a)_b$ is the inverse of the remainder of the division of $-a$ by $-b$ (because $(a)_b = -(-a)_{-b}$).

Therefore $a \equiv c \pmod{b}$ if and only if $(a)_b = (c)_b$.

2.4 Some analytic properties of b-parts and asymptotic formulas

In this section we consider b -parts functions and give some asymptotic formulas for summation and mean values of various-decimal part functions.

Recall that for every positive integer n , $d(n)$ and $\sigma(n)$ denote the number and sum of the all divisors of n respectively. The arithmetical function Λ is called Mangoldt's function and defined by

$$\Lambda(n) = \begin{cases} \log p & \text{if } n = p^m \text{ for some prime } p \text{ and } m \geq 1 \\ 0 & \text{otherwise} \end{cases}$$

If f is an arithmetical function and $x > 0$, then the partial summation of f is defined by

$$\sum_{k \leq x} f(k) = \begin{cases} \sum_{k=1}^{[x]} f(k) & x \geq 1 \\ 0 & 0 < x < 1 \end{cases}$$

Also $\tilde{f}(x) = \frac{1}{x} \sum_{k \leq x} f(k)$ is the mean value function of f .

The function ζ given by

$$\zeta(s) = \begin{cases} \sum_{n=1}^{\infty} \frac{1}{n^s} & s > 1 \\ \lim_{x \rightarrow \infty} \left(\sum_{n \leq x} \frac{1}{n^s} - \frac{x^{1-s}}{1-s} \right) & 0 < s < 1 \end{cases}$$

denotes the Riemann zeta function.

If $g(x) > 0$ for all $x \geq a$, we write $f(x) = O(g(x))$ to mean that there exists a constant $M > 0$ such that $|f(x)| \leq M g(x)$ for all $x \geq a$. Also an equation of the form $f(x) = h(x) + O(g(x))$ means that $f(x) - h(x) = O(g(x))$. We say that $f(x)$ is asymptotic to $g(x)$ as $x \rightarrow \infty$ and write $f(x) \sim g(x)$ as $x \rightarrow \infty$ if $\lim_{x \rightarrow \infty} \frac{f(x)}{g(x)} = 1$.

Now we want to study $\sum_{n \leq x} (x)_n$. Note that if $x = m$ is a positive integer, then this is the sum of remainders of divisions of m by positive integers less than or equal to m . The following statement is the well known formula in analytic number theory (Theorem 3.11 in [1]).

Theorem. If $F(x) = \sum_{n \leq x} f(n)$ we have

$$\sum_{n \leq x} \sum_{d|n} f(d) = \sum_{n \leq x} f(n) \left[\frac{x}{n} \right] = \sum_{n \leq x} F\left(\frac{x}{n}\right), \quad (2.4.1)$$

Putting $f = \iota_{\mathbb{R}}$ (the identity function) in the above equation we get

$$\sum_{n \leq x} [x]_n = \sum_{n \leq x} \sigma(n) = \frac{1}{2} \sum_{n \leq x} \left[\frac{x}{n} \right]^2 + \left[\frac{x}{n} \right]. \quad (2.4.2)$$

On the other hand

$$\sum_{n \leq x} (x)_n = \sum_{n \leq x} x - [x]_n = x[x] - \sum_{n \leq x} [x]_n,$$

so

$$\sum_{n \leq x} (x)_n = x[x] - \sum_{n \leq x} \sigma(n) = x[x] - \frac{1}{2} \sum_{n \leq x} \left[\frac{x}{n} \right]^2 + \left[\frac{x}{n} \right]. \quad (2.4.3)$$

This formula connects the partial summation to partial summation of σ . For more explanation we first introduce the b -parts and various-decimal parts functions.

Definition 2.4.1. Fix real numbers a and $b \neq 0$. By the notations $(-)_b$ and $(a)_-$ $[-]_b$ and $[a]_-$ we mean b -decimal part and various-decimal part- a [b -integer part and various-integer part- a] functions, respectively and defined by

$$(-)_b(x) = (x)_b, \quad (a)_-(x) = (a)_x, \quad [-]_b(x) = [x]_b, \quad [a]_-(x) = [a]_x.$$

We call both $(-)_b$ and $[-]_b$, b -parts functions and often these notations are replaced by $()_b$ and $[]_b$. Also the functions $(a)_-$ and $[a]_-$ are called various-parts functions (of a).

The domains of b -parts functions are \mathbb{R} , but the various-parts functions cannot be defined at zero by the equations. But considering $\lim_{x \rightarrow 0} (a)_x = 0$ and $\lim_{x \rightarrow 0} [a]_x = a$, we can extend their definitions to all real numbers. So

$$(a)_-(x) = \begin{cases} (a)_x & x \neq 0 \\ 0 & x = 0 \end{cases}, \quad [a]_-(x) = \begin{cases} [a]_x & x \neq 0 \\ a & x = 0 \end{cases}$$

Hence the various-parts functions are continuous at zero.

It is easy to see that the following properties hold:

(A) For every $a \in \mathbb{R}$ and $b \in \mathbb{R} \setminus \{0\}$ we have $(a)_- + [a]_- = a$, $(\)_b + [\]_b = \iota_{\mathbb{R}}$ (the identity function on \mathbb{R}). The b -parts functions satisfy the following equations

$$f(f(x) + y - f(y)) = f(x) : \text{ } b\text{-parts type functional equation}$$

and $(\)_b$ also satisfies the equations

$$f(x + y - f(y)) = f(x) : \text{ strong } b\text{-parts type functional equation.}$$

$$f(f(x + y) + z) = f(x + f(y + z)) = f(x + y + z) : \text{ strongly associative equation.}$$

(B) The b -decimal part function is bounded ($|(\)_b| < |b|$), b -periodic, idempotent and we have $(\)_b o [\]_b = [\]_b o (\)_b = 0$. These functions are totally disconnected at every points of $b\mathbb{Z} = \langle b \rangle$. But the set of all disconnected points of $(a)_-$ and $[a]_-$ is $\text{Zero}((a)_-) = \{\pm a, \pm \frac{a}{2}, \dots\}$, where $\text{Zero}((a)_-)$ is the set of all zeros of $(a)_-$. The main properties of various-parts functions of a occurs on the open interval $(-|a|, |a|)$.

If $a > 0$, then

$$(a)_-(x) = \begin{cases} a+x & x \leq -a \\ a+2x & -a < x \leq -\frac{a}{2} \\ \vdots & \vdots \\ a+nx & -\frac{a}{n-1} < x \leq -\frac{a}{n} \\ \vdots & \vdots \\ 0 & x = 0 \\ \vdots & \vdots \\ a-nx & \frac{a}{n+1} < x \leq \frac{a}{n} \\ \vdots & \vdots \\ a-x & \frac{a}{2} < x \leq a \\ a & a < x \end{cases}$$

Also if $a, b > 0$, then we have the following limit properties:

$$\lim_{x \rightarrow 0^+} (x)_b = 0, \quad \lim_{x \rightarrow 0^-} (x)_b = b, \quad \lim_{x \rightarrow -\infty} (a)_x = -\infty; \quad \lim_{x \rightarrow +\infty} (a)_x = a,$$

also the limits of $(x)_b$ at $\pm\infty$ do not exist.

(C) If $b > 0$, then the function $()_b$ corresponds every x to the remainder of its division by b . For the three other functions there are similar explanations, considering the generalized division algorithm.

If a is an integer, then $\text{Zero}((a)_-) \cap \mathbb{Z} = D(a)$ (i.e. the set of all divisors of a).

Now we come back to the equation (2.4.3). It implies that

$$\sum_{n \leq a} ((a)_- + \sigma)(n) = \sum_{n \leq a} (a)_n + \sum_{n \leq a} \sigma(n) = a[a],$$

so the mean value function of $(a)_- + \sigma$ at a is equal to the integer part of a .

Now we are ready to introduce the asymptotic formulas for partial sums and mean values of various-parts functions, and also determine their average orders.

Theorem 2.4.1. *If $x \geq 1$, then*

(a)

$$\sum_{n \leq x} [x]_n = \sum_{n \leq x} \sigma(n) = \frac{\pi^2}{12} x^2 + O(x \log(x)),$$

$$\sum_{n \leq x} (x)_n = x[x] - \frac{\pi^2}{12} x^2 + O(x \log(x)).$$

(b) *If x is fixed and $f(t) = (x)_t$, for every $t > 0$, then $(\tilde{f}(t) = \frac{1}{t} \sum_{n \leq t} (x)_n$ and) we have $\tilde{f}(x) = [x] - \frac{\pi^2}{12} x + O(\log(x))$ and so*

$$\tilde{f}(x) \sim [x] - \frac{\pi^2}{12} x \quad \text{as } x \rightarrow \infty.$$

(c) *If $x = N$ is a fixed positive integer, then the average order of $(N)_n$ ($\frac{1}{N} \sum_{n \leq N} (N)_n$) is $\frac{12-\pi^2}{12} N \approx 0.1775 N$.*

(d)

$$\sum_{n \leq x} \frac{\Lambda(n)}{n} (x)_n = x + O(x + \log(x)).$$

(e)

$$\sum_{p \leq x} (x)_p \log \sqrt[p]{p} = O(x),$$

where the sum is extended over all primes.

Proof. Considering (2.4.2), (2.4.3) and the following well known asymptotic formulas for the partial sum of σ (see Theorem 3.4 in [1]), we have

$$\sum_{n \leq x} \sigma(n) = \frac{1}{2} \zeta(2) x^2 + O(x \log(x)),$$

and since $\zeta(2) = \frac{\pi^2}{6}$ we get (a). The part (b) is concluded from (a), considering $\lim_{x \rightarrow \infty} \frac{\log(x)}{[x] - \frac{\pi^2}{12} x} = 0$, and (b) implies (c) directly. Now applying the following asymptotic formula of Theorem 3.16 in [1]:

$$\sum_{p \leq x} \left[\frac{x}{p} \right] \log p = x \log x + O(x),$$

we get

$$\sum_{p \leq x} (x)_p \frac{\log p}{p} = x \left(\sum_{p \leq x} \frac{\log p}{p} - \log(x) \right) + O(x).$$

On the other hand $\sum_{p \leq x} \frac{\log p}{p} = \log(x) + O(1)$ (see Theorem 4.10 in [1]). These equations imply (e). The following equations (see [1])

$$\sum_{n \leq x} \frac{\Lambda(n)}{n} = \log(x) + O(1) \quad , \quad \sum_{n \leq x} \Lambda(n) \left[\frac{x}{n} \right] = x \log(x) + O(x),$$

imply (d) similar to (e).

□

Chapter 3

Finite and Infinite b -Representation of Real Numbers

In this chapter we first consider b -digital sequences. Thereafter by using b -parts and b -digital sequences we not only give a new proof for the unique infinite b -adic representation of real numbers, based on an arbitrary integer $b \neq 0, \pm 1$, but also provide several direct formulas for their digits. In this way we prove necessary and sufficient conditions for a (two sided) sequence to be digital sequence. In continuation we extend the base b from integers to real numbers and introduce finite b -representation of real numbers based on an arbitrary real number $b \neq 0, \pm 1$, by using b -parts and the generalized division algorithm. Finally applying b -parts we describe and consider uniform distribution of real sequences modulo an arbitrary fixed real number $b \neq 0$ and state its relation to the uniform distribution of real sequences modulo 1.

3.1 Introduction

In mathematical numeral systems, the (integer) base or radix is usually the number of unique digits, including zero, that a numeral system uses to represent numbers. For

example, for the decimal system (the most common system in use today) the radix is 10, because it uses the 10 digits from 0 through 9.

A radix-representation in \mathbb{R} is started with elementary facts. If $b > 1$ is a fixed positive integer, then for any given positive integer a , there exists a nonnegative integer N and $N + 1$ integers a_0, a_1, \dots, a_N such that a may be represented uniquely in the following form

$$a = a_N b^N + a_{N-1} b^{N-1} + \dots + a_0 \quad \text{with } 0 \leq a_n < b \text{ for } 0 \leq n \leq N. \quad (3.1.1)$$

In general, for a given positive integer b , called the radix or base, a set D of real numbers with the property that every $r \in \mathbb{R}$ can be represented in the form $a = \sum_{i=-N(a)}^{\infty} a_i b^{-i}$, $a_i \in D$, is called a radix representation for \mathbb{R} . The standard representation is to use $b = 10$ and $D = \{0, 1, 2, \dots, 8, 9\}$.

The most common radix representation of real numbers is b -adic expansion (see [4]), that is, an expansion of the form

$$a = [a] + \sum_{n=1}^{\infty} \frac{a_n}{b^n} = [a].a_1 a_2 \dots a_n \dots \quad (3.1.2)$$

where the "digits" a_n are integers with $0 \leq a_n < b$ for $n \geq 1$, and also $a_n < b - 1$ for infinitely many n .

But we have several kinds of representation for real numbers. For a given base, every representation corresponds to exactly one real number. A representation can be infinite or finite, unique or non-unique, digital or non-digital (see [2, 4, 5, 8, 22]). For example representation of real numbers by some functions (f-representation) or continued fractions are different from the radix representation and may be not unique or digital.

In [21] some applications of b -parts for the infinite radix representation of real numbers

(to the base integer $b \neq 0, \pm 1$) were studied.

3.2 b -Digital sequences and their relations to b -parts

We call a function $a : \mathbb{Z} \rightarrow S$ (where $S \neq \emptyset$ is an arbitrary set) a "two sided sequence" and denote it by $\{a_n\}_{+\infty}^{-\infty}$. Since infinite radix representation of a real number is in fact a two sided sequence of integers with some properties, and considering the fact that it is possible that a real number has two different radix representation to the base a fixed integer $b > 1$ (e.g. $\frac{51}{10} = 5.1000 \dots = 5.0999 \dots$), we should choose only one of them as the unique representation (like b -adic representation).

Definition 3.2.1. Let $b > 1$ be a fixed positive integer. A b -digital sequence (to base b) is a two-sided sequence $\{a_n\}_{+\infty}^{-\infty}$ of integers which satisfy the following conditions

- i) $0 \leq a_n < b : \forall n \in \mathbb{Z}$,
- ii) there exists an integer N such that $a_n = 0$, for all $n > N$
- iii) for every integer m , there exists an integer $n \leq m$ such that $a_n \neq b - 1$.

In the above definition we will consider N as the largest integer that $a_N \neq 0$, when a_n is not the zero b -digital sequence (we set $N = 0$, for the zero b -digital sequence). The following theorem not only characterizes the b -digital sequences but also gives us a new proof for the b -adic representation of positive real numbers by using b -parts.

Theorem 3.2.1. (*Fundamental theorem of b -digital sequences*). Let $b > 1$ be a positive integer. A two-sided sequence $\{a_n\}_{+\infty}^{-\infty}$ of integers is a b -digital sequence if and only if there exists a nonnegative real number a such that

$$a_n = ([b^{-n}a])_b : \forall n \in \mathbb{Z}.$$

Proof. Let $\{a_n\}_{N,b}$ be a b -digital sequence. Clearly the series $\sum_{+\infty}^{-\infty} a_n b^n$ is convergent (and equal to $\sum_N^{-\infty} a_n b^n = a_N b^N + a_{N-1} b^{N-1} + \dots$). Put $a = \sum_{+\infty}^{-\infty} a_n b^n$. If m is a

fixed integer, then

$$b^{-m}a = \sum_{+\infty}^{-\infty} a_n b^{n-m} = \sum_{+\infty}^{-\infty} a_{n+m} b^n.$$

There exists an integer $k \leq -1$ such that $a_{k+m} \leq b-2$ so,

$$\begin{aligned} 0 &\leq \sum_{-1}^{-\infty} a_{n+m} b^n \leq -b^k + \sum_{-1}^{-\infty} (b-1)b^n \\ &= 1 - b^k < 1, \end{aligned}$$

therefore,

$$[b^{-m}a] = \sum_{+\infty}^0 a_{n+m} b^n, \quad (b^{-m}a) = \sum_{-1}^{-\infty} a_{n+m} b^n.$$

Thus

$$\begin{aligned} [b^{-m}a]_b &= b[b^{-m-1}a]_1 = b \sum_{+\infty}^0 a_{n+m+1} b^n \\ &= \sum_{+\infty}^1 a_{n+m} b^n. \end{aligned}$$

Hence

$$a_m = [b^{-m}a] - [b^{-m}a]_b = (b^{-m}a)_b - (b^{-m}a)_1. \quad (3.2.1)$$

Now considering (2.2.2) and (3.2.1) we get

$$\begin{aligned} a_n &= ([b^{-n}a])_b = [b^{-n}a] - [b^{-n}a]_b \\ &= (b^{-n}a)_b - (b^{-n}a)_1 : \quad \forall n \in \mathbb{Z}. \end{aligned} \quad (3.2.2)$$

Conversely, let $a \geq 0$ and $a_n = ([b^{-n}a])_b$, for all integer n . Clearly a_n satisfies the conditions (i) and (ii) of Definition 3.2.1. If N_1, N_2 are integers such that $N_1 \geq N_2$, then

$$\begin{aligned} \sum_{N_1}^{N_2} a_n b^n &= \sum_{N_1}^{N_2} \{[b^{-n}a]_1 b^n - [b^{-n-1}a]_1 b^{n+1}\} \\ &= [b^{-N_2}a]_1 b^{N_2} - [b^{-N_1-1}a]_1 b^{N_1+1}. \end{aligned} \quad (3.2.3)$$

Now if there exists an integer m such that $a_n = b-1$ for all $n \leq m$, then

$$\sum_m^{-\infty} a_n b^n = \sum_m^{-\infty} (b-1)b^n = b^{m+1}.$$